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FOLIATIONS WITH INTEGRABLE TRANSVERSE G-STRUCTURES

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Introduction

The existence of integrable G-structures on a smooth manifold M^n has been examined by Grove and Hansen [7]. An integrable G-structure on M^n means that we can find an atlas on M^n such that the Jacobean of the transition function in any intersectiion belong to G, when G is a discrete subgroup of $Gl(n, R)$ (if G is finite or more generally if G belongs to some compact subgroup we can assume $G \subset 0(n)$). Equivalently we could require M^n to have an affine flat structure or a Riemannian flat structure in the case $G \subset 0(n)$ and holonomy group contained in G.

In this paper we want to generalize this. Suppose we have a smooth manifold M^n with a smooth foliation $\mathfrak{F}$. An integrable transverse G-structure (for short, a $\Gamma(G)$-foliation) means that we can find an atlas on M^n such that the Jacobean of the transition function transverse to $\mathfrak{F}$ in any intersection belong to G. Equivalently we could require M^n and $\mathfrak{F}$ to have a basic connection which is affine flat or Riemannian flat in the case $G \subset 0(q)$ and holonomy group contained in G.

To avoid confusion with other notation, it should be mentioned that a transverse G-structure as defined by Conlon [4] is a weaker notion (except for codimension 1). A transverse G-structure is the same as giving a basic connection with holonomy group contained in G.—So much for notation, and we want to show

Theorem A (*Theorems* 1.4 *and* 2.1). *If M^n is open, then M^n has a foliation of codimension q with an integrable transverse e-structure if and only if M^n has a trivial q-dimensional subbundle of the tangent-bundle. If M^n is closed, then M^n has a foliation of codimension q with an integrable transverse e-structure if and only if M^n is a fiber bundle*

$$F \to M^n \to T^q$$

with a torus as base.

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Theorem B (*Theorem* 3.1). *If M^n has a foliation with an integrable transverse G-structure, then there is a principal G-bundle π: $\overline{M}^n \to M^n$ such that $\overline{M}^n$ has an integrable transverse e-structure.*

Theorem C (*Theorems* 2.4, 2.5 *and* 2.6). *If M^n is closed and has an integrable transverse e-structure, then all leaves L are diffeomorphic, and we have a Serre-fibration*

$$L \to M^n \to T^s \quad (s \geqslant q),$$

with a torus as base.

Theorem D (*Theorem* 3.4). *If M^n is closed and has an integrable transverse G-structure with $G \subset 0(q)$, then the structure can be reduced to an integrable transverse G'-structure with $G' \subset G$ and G' finite.*

1. Quantitative theory for $\Gamma(G)$-foliations

Let M^n be an n-dimensional smooth manifold with a foliation of dimension p and codimension q $(p + q = n)$. $\mathfrak{F}$ is given by a maximal atlas $\{I_i^p \times I_i^q\}$ such that the transition functions

$$I_i^p \times I_i^q \to I_j^p \times I_j^q$$

are of the form

$$(x, y) \to (\psi(x, y), \varphi(y)).$$

Suppose we are given a discrete subgroup $G \subset Gl(q, R)$. Let $\Gamma q(G)$ be the group of diffeomorphisms of R^q, which has a Jacobean belonging to G. Then we have a fibration of groups

$$R^q \to \Gamma q(G) \to G.$$

The last map is the Jacobean, and the first map is the inclusion of the discrete subgroup of translations in R^q.

Definition 1.1. If a subatlas can be chosen such that $\varphi \in \Gamma q(G)$ or equivalently such that $\mathrm{Jac}(\varphi) \in G$, then $\mathfrak{F}$ is said to be a $\Gamma q(G)$-foliation, which is just a shorter notation for an integrable transverse G-structure.

Lemma 1.2. *$\mathfrak{F}$ is a $\Gamma q(e)$-foliation $\Leftrightarrow$ $\mathfrak{F}$ is given by a closed globally decomposable q-form Ω.*

Remark. This is a special case of volume-preserving foliations, in which case Ω is only assumed to be locally decomposable.

Proof. Suppose $\mathfrak{F}$ is a $\Gamma q(e)$-foliation. Then Ω is given by pull-back of the standard q-form on R^q.

Suppose $\mathfrak{F}$ is given by $\Omega = \omega_1 \wedge \cdots \wedge \omega_q$. Then locally $\omega_i = df_i$, and $(f_1, \cdots, f_q) \in R^q$ defines a chart of the type $I_i^p \times I_i^q$. But $(f_1, \cdots, f_q)$ is uniquely defined up to constants, so a different choice would give $\omega_i = dg_i$, and $(g_1,\cdots,g_q) \in R^q$ defines another chart of the type $I_j^p \times I_j^q$. Since $g_i = f_i + c_i$, the transition function

$$I_i^p \times I_i^q \to I_j^p \times I_j^q$$

is given by

$$(x, y) \to (\psi(x, y), y + c) \quad \text{with } c = (c_1,\cdots,c_q).$$

Before stating the next proposition let us define some standard terminology.

Definitions. Let $\tau(\mathcal{F})$ be the tangent bundle to the foliation, and $\gamma(\mathcal{F})$ the normal bundle. Then $T(M^n) \cong \tau(\mathcal{F}) \oplus \gamma(\mathcal{F})$ where $T(M^n)$ denotes the tangent bundle of M^n, and we choose some Riemannian metric to identify $\gamma(\mathcal{F})$ as a subbundle of $T(M^n)$.

A connection ∇ on $\gamma(\mathcal{F})$ is said to be basic if whenever X is a section of $\tau(\mathcal{F})$, and Y is a section of $\gamma(\mathcal{F})$, which is parallel along the leaves, then $\nabla_X Y \equiv 0$.

An affine flat connection ∇ is one whose curvature tensor and torsion tensor vanish, i.e.,

$$\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \equiv 0 \quad \forall X, Y \in T(M^n),$$
$$\nabla_X Y - \nabla_Y X - [X, Y]_{\gamma(\mathcal{F})} \equiv 0 \quad \forall X, Y \in \gamma(\mathcal{F}).$$

Proposition 1.3. *Let $G \subset Gl(q, R)$ be a discrete subgroup. Then M^n has a $\Gamma(G)$-foliation if and only if M^n has a foliation $\mathcal{F}$ with a basic connection on $\gamma(\mathcal{F})$ which is affine flat (Riemannian flat in the case $G \subset 0(q)$) and with holonomy group contained in G.*

Proof. Suppose $\gamma(\mathcal{F})$ has a basic connection ∇ which is affine flat. Since the curvature is zero, $\gamma(\mathcal{F})$ has a local basis of parallel fields E_α $(\alpha = 1,\cdots,q)$; since the torsion is zero, we have

$$\forall\alpha,\beta: \quad \nabla_{E_\alpha}E_\beta - \nabla_{E_\beta}E_\alpha - \left[E_\alpha, E_\beta\right]_{\gamma(\mathcal{F})} = 0,$$
$$\Downarrow$$
$$\forall\alpha,\beta: \quad \left[E_\alpha, E_\beta\right] \in \tau(\mathcal{F}).$$

By the usual argument we can find E'_α $(\alpha = 1,\cdots,q)$ such that $E'_\alpha = E_\alpha$ (mod $\tau(\mathcal{F})$) and $\forall_{\alpha,\beta}: [E'_\alpha, E'_\beta] = 0$. This means that $E'_1,\cdots,E'_q$ are local coordinate vector-fields defining a chart

$$I_i^p \times I_i^q$$

such that the connection ∇ is induced from the affine structure in I_i^q. Transition functions between two of these charts

$$\varphi: I_i^p \times I_i^q \to I_j^p \times I_j^q$$

has to preserve the affine structure on the second factor, hence the holonomy group is some discrete group G.

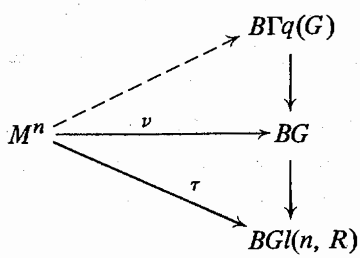
On the other hand suppose we have a $\Gamma(G)$-foliation with some charts $\{(I_i^p \times I_i^q)\}_{i\in I}$. Define a connection on these charts induced from the affine structure on I_i^q. We must show that the connections agree on the intersection, which we do by showing that parallelism is the same in both coordinate systems.

A vector field X is parallel in $I_i^p \times I_i^q$ if and only if the projection X_i to I_i^q is constant and parallel in $I_j^p \times I_j^q$ if and only if the projection X_j to I_j^q is constant. Now X_j is related to X_i by

$$X_j = D(\varphi)X_i.$$

But $D(\varphi)$ is a fixed linear map, so X_j is constant if and only if X_i is constant.

Theorem 1.4 (Gromov [6], Haefliger [8], Phillips [11]). *Integrable homotopy classes of $\Gamma q(G)$-foliations on an open manifold M^n is in* 1-1 *correspondence with homotopy classes of mappings (dotted arrow) making the following diagram commutative*:



Together with homotopy classes of subbundles of $T(M^n)$ (tangent-bundle classified by τ) given by v.

Proof. This is a general result which is valid for all pseudogroups Γq.

Corollary 1.5. *A necessary and sufficient condition for the existence of a $\Gamma q(e)$-foliation is that the subbundle defined by v be trivial.*

Corollary 1.6. *Integrable homotopy classes of $\Gamma q(e)$-foliations are in* 1-1 *correspondence with $(H^1(M^n, R))^q$ together with homotopy classes of trivial q-dimensional subbundles of the tangent bundle. The $\Gamma q(e)$-foliations corresponding to the trivial cohomology class are induced by a submersion to R^q. The others are not.*

The quantitative theory for closed manifolds is rather complicated compared to that for open manifolds. This is contrary for the qualitative theory as we shall see in the next section.

2. Qualitative theory for $\Gamma(e)$-foliations on closed manifolds

Theorem 2.1. *If M^n is closed with $\Gamma q(e)$-foliation $\mathfrak{F}$, then M^n is a fiberbundle*

$$F \to M^n \to T^q$$

which can be chosen such that any leaf L in $\mathfrak{F}$ is a covering of F. Furthermore $L \times R^q$ is a covering of M^n.

Proof. Suppose $\mathfrak{F}$ is given by $\Omega = \omega_1 \wedge \cdots \wedge \omega_q$ (compare Lemma 1.2). By the theory of harmonic 1-forms (see [14]) we can approximate ω_i by a closed nonsingular 1-form $\Sigma_{j=1}^{p_i} q_{ij} f_{ij} * (d\theta)$ where f_{ij}: $M^n \to S^1$ and $q_{ij} \in Q$. But then also some 1-form $\Sigma_{j=1}^{p_i} n_{ij} f_{ij} * (d\theta)$ is nonsingular ($n_{ij} \in Z$) which means that $\Pi_{i=1}^{q} \Sigma_{j=1}^{p_i} n_{ij} f_{ij}$: $M^n \to T^q$ is a submersion.

Since M^n is compact, this gives the fiber bundle. Take the covering $\tilde{M}^n$ of M^n such that $\pi_1(F) \cong \pi_1(\tilde{M}^n)$. Then

$$\tilde{M}^n = F \times R^q$$

The foliation $\mathfrak{F}$ lifts to a foliation $\tilde{\mathfrak{F}}$ transverse to the R^q-factor, since the original 1-forms are close to the new (rational) 1-forms. The leaves L in $\mathfrak{F}$ lifts to leaves in $\tilde{\mathfrak{F}}$ which are diffeomorphic, since nothing in $\pi_1(L)$ is killed by the lifting. This means that L is a covering of F.

Then take the covering $\tilde{M}^n$ of M^n such that $\pi_1(L) \cong \pi_1(\tilde{M}^n)$. Thus

$$\tilde{M}^n = L \times R^q$$

is similar to the above. But we can do more, and will see that we have a Serre fibration

$$L \to M^n \to T^s.$$

This is no longer a locally trivial fibration as in Theorem 2.1, and we have $s \geqslant q$.

To start let us recall the definition of a simplicial set (see Kan [9]).

Definition 2.2. A simplicial set K is a sequence of sets

$$K = \{K_0, K_1, \cdots, K_n, \cdots\}$$

together with functions (called face and degeneracy operators respectively) for each $0 \leqslant i \leqslant n$:

$$d_i: K_n \to K_{n-1},$$
$$s_i: K_n \to K_{n+1}.$$

The functions are required to satisfy certain identities. A simplicial map f: $K \to L$ is a family of functions f_n: $K_n \to L_n$ commuting with d_i and s_i. Elements $x \in K_n$ are called n-simplices, and elements $x \in K_0$ are also called vertices.

Let $\Delta[n]$ be the standard n-simplex, which is the simplicial set with vertices $0, 1, \cdots, n$ and q-simplices

$$\Delta[n]_q = \{\langle v_0, \cdots, v_q \rangle \mid 0 \leqslant v_0 \leqslant \cdots \leqslant v_q \leqslant n\},$$
$$d_i\langle v_0, \cdots, v_q\rangle = \langle v_0, \cdots, \hat{v}_i, \cdots, v_q\rangle,$$
$$s_i\langle v_0, \cdots, v_q\rangle = \langle v_0, \cdots, v_i, v_i, \cdots, v_q\rangle.$$

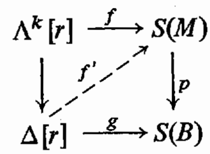
$\Delta[n]$ is generated by the n-simplex

$$i_n = \langle 0, 1, \cdots, n\rangle \in \Delta[n]_n.$$

Let $\Lambda^k[n]$ be the sub-simplicial set generated by all $d_i(i_n)$ for $i \neq k$.

Remark. Let K be any simplicial set, and $x \in K_n$ any n-simplex. Then there is a unique simplicial map (the representative map for x) $fx: \Delta[n] \to K$ such that $fx(i_n) = x$.

Definition 2.3. A surjective simplicial map $p: E \to B$ is called a Kan fibration if whenever $f: \Lambda^k[n] \to E$ and $g: \Delta[n] \to B$ with $p \circ f = g \mid \Lambda^k[n]$, then there is an extension of f to a map $f': \Delta[n] \to E$ with $p \circ f' = g$. The picture for this is the diagram



Call f' an extension of f which covers g, and $p^{-1}(*)$ the fiber.

Now consider a closed manifold M^n with a $\Gamma q(e)$-foliation. Let $S(M)$ be the simplicial set with $S(M)_r = \{\text{smooth maps } \Delta^r \to M^n\}$ with Δ^r denoting the geometric r-simplex. Let $S(B)$ be the simplicial set with

$$S(B)_r = S(M)_r/\sim\ ,$$

where two simplices are equivalent if one can be translated onto the other by holonomy projection along leaves. This is the equivalence relation generated by the following relation:

Take one of the charts $I_i^p \times I_i^q$. Then any two smooth maps $\Delta^r \to I_i^p \times I_i^q$ are said to be equivalent if they agree on the second factor.

In general two simplices are said to be equivalent if they are equivalent on some subdivision.

Suppose now that we have a simplex $f: \Delta^r \to M^n$ with $f(v_0) = x$. Then for any x' belonging to the same leaf as x we can choose a simplex $f': \Delta^r \to M^n$ equivalent to f such that $f'(v_0) = x'$, and this is unique in the sense that if two simplices belong to the same chart $I_i^p \times I_i^q$, then they agree on the second factor if they agree on one point.

This is done by taking an appropriate subdivision of a path from x to x' on the leaf, and a subdivision of $f\colon \Delta^r \to M^n$. Then we can extend over each little path-segment and each little simplex at a time. Using the compactness of M^n and the usual argument with Lebesgue numbers, this process will never stop. In fact we can find a cover by charts $\{I_i^p \times I_i^q\}$ such that any little path-segment of length less than ε and any simplex can be extended such that the length of all the new paths on leaves have a length less than ε. Furthermore look at simplices in a chart $I_i^p \times I_i^q$. Then we can take the composite

$$\Delta^r \to I_i^p \times I_i^q \to I_i^q \subset R^q,$$

and the image in R^q is well defined up to translation, which gives the uniqueness.

Now find a vertex $*$ in $S(B)_0$. Then we have a natural surjective map

$$p\colon S(M) \to S(B)$$

with fiber $p^{-1}(*) = S(L)$, where L is the leaf corresponding to $*$, and $S(L)$ is the simplicial set with

$$S(L)_r = \{\text{smooth maps } \Delta^r \to L\}.$$

Theorem 2.4 (see also Godbillon [5]).

$$S(L) \to S(M) \to S(B)$$

is a Kan-fibration.

Proof. We are given the following diagram

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{f} & E \\ \downarrow & \overset{f'}{\nearrow} & \downarrow p \\ \Delta[n] & \xrightarrow{g} & B \end{array}$$

and we want to find f'. This means that we have two smooth maps

$$f\colon \Lambda^r \to M^n, \quad g\colon \Delta^r \to M^n,$$

where Λ^r is the boundary less one face of the geometric r-simplex Δ^r, and they agree on Λ^r up to equivalence. Then use this equivalence on Λ^r to translate all of Δ^r (by holonomy projection) to a smooth map

$$f'\colon \Delta^r \to M^n$$

which agree with f on Λ^r.

In the process we are of course using the existence and uniqueness property for holonomy projections established above.

It follows from the general theory of Kan-fibrations (Zisman [17]) (or from a similar result for Serre fibrations (Steenrod [13])) that two leaves L_1 and L_2

have the same weak homotopy type, in particular that they are both compact or both noncompact, but in this particular case we further have

Theorem 2.5. *Two leaves L_1 and L_2 are diffeomorphic.*

Proof. We use the Kan-fibration property for 1-simplices. In this case Δ^1 is a geometric 1-simplex, and Λ^1 is an endpoint. So we choose a smooth map $f: [0, 1] \to M^n$ with $f(0) \in L_1$ and $f(1) \in L_2$. We are also given another point $x \in L_1$. Then by the Kan-fibration property there is a smooth map $f': [0, 1] \to M^n$ with $f'(0) = x$, and $\forall t \in [0, 1]$, $f'(t)$ belongs to the same leaf as $f(t)$. In particular we have $f'(1) \in L_2$. Now we can choose f' uniquely by demanding that f' is tangent to some fixed normal bundle to the foliation $\mathcal{F}$. The map $x \to f'(x)$ gives the desired diffeomorphism. Again by the general theory of Kan-fibrations we have a long-exact homotopy sequence

$$\cdots \to \pi_i(L) \to \pi_i(M^n) \to \pi_i(B) \to \pi_{i-1}(L) \to \cdots$$

and a spectral sequence $(E^r_{p,q}, d^r)$ with

$$E^2_{p,q} = H_p\big(B;\, H_q(L)\big) \text{ (local coefficients)}$$

and converging to $H_*(M^n)$. There is an analogous for cohomology. If the local coefficient system is constant we have from Serre [12] that B is of finite type in homology, but in this particular case we further have

Theorem 2.6. *$B = |S(B)|$ is homotopy equivalent to a torus.*

Proof. Suppose the foliation $\mathcal{F}$ is given by $\Omega = \omega_1 \wedge \cdots \wedge \omega_q$, then there is a smooth map $t: R^q \to M^n$ such that

$$t^*\big(\omega_1 \wedge \cdots \wedge \omega_q k\big) = dx^1 \wedge \cdots \wedge dx^q.$$

The set where t is defined is both open and closed since M^n is compact, hence t is defined on all of R^q. By the same type of argument t intersects all leaves in $\mathcal{F}$. There is an induced equivalence relation on $S(R^q)$, and it is clear that

$$S(R^q)/\sim = S(M^n)/\sim = S(B),$$

where $S(R^q)$ is the simplicial set with

$$S(R^q)_r = \{\text{smooth maps } \Delta^r \to R^q\}.$$

Furthermore as we had already seen when we studied uniqueness of holonomy projections, equivalent simplices

$$\Delta^r \to R^q$$

have a well-defined image in R^q up to translation. Thus the equivalence relation $\sim$ on $S(R^q)$ is induced by a subgroup $Z^s \subset R^q$ of translations, and hence

$$B = |S(R^q)/\sim| = T^s.$$

Example. Suppose we have a foliation $\mathcal{F}$ on M^n such that for all leaves L we have

$$L \cong S^{2k} \times R^l \quad (k > 0).$$

Then using the long-exact homotopy sequence we get immediately that

$$\pi_*(M^n) \cong \pi_*(S^{2k} \times T^{n-2k}).$$

Or assuming that the local coefficient system is constant, from the spectral sequence we get a Gysin-sequence (see Serre [12]):

$$\cdots \to H^{i+2k}(M^n) \to H^i(B) \xrightarrow{\chi} H^{i+2k+1}(B) \to H^{i+2k+1}(M^n) \to \cdots.$$

The middle map is multiplication by the characteristic class $\chi \in H^{2k+1}(B)$, which is 2-torsion and hence zero since $B = T^{n-2k}$. So we get

$$H^*(M^n) \cong H^*(S^{2k} \times T^{n-2k}).$$

3. Qualitative theory for $\Gamma(G)$-foliations and concluding remarks about reduction to finite structures

Theorem 3.1. *Suppose M^n has a $\Gamma q(G)$-foliation. Then there exists a principal G-bundle π: $\overline{M}^n \to M^n$ such that $\overline{M}^n$ has a $\Gamma q(e)$-foliation.*

Proof. Let $\{I_i^p \times I_i^q\}$ be an atlas on M^n giving the $\Gamma q(G)$-foliation. Now we want to construct a manifold $\overline{M}^n$ such that $\{I_i^p \times I_i^q \times G\}$ is an atlas on $\overline{M}^n$ giving a $\Gamma q(e)$-foliation. Define

$$\overline{M}^n = \amalg(I_i^p \times I_i^q \times G)/\sim$$

where the relation $\sim$ is defined as follows:

$$(x, y, g) \sim (\psi(x, y), \varphi(y), \mathrm{Jac}(\varphi) \cdot g).$$

Here we assume that

$$M^n = \amalg(I_i^p \times I_i^q)/\sim$$

with the relation $\sim$ defined as follows:

$$(x, y) \sim (\psi(x, y), \varphi(y)).$$

Give $\overline{M}^n$ the quotient topology, and it is not difficult to see that the natural projection π: $\overline{M}^n \to M^n$ is a principal G-bundle. Now define new charts by

$$(x, y, g) \to (x, g^{-1}(y)).$$

From which it follows that there is a new chart

$$(\psi(x, y), \varphi(y), \mathrm{Jac}(\varphi) \cdot g) \to (\psi(x, y), g^{-1}\mathrm{Jac}(\varphi^{-1})\varphi(y)).$$

In those new charts, the transition function is the following on second factor:

$$g^{-1}(y) \to g^{-1}\mathrm{Jac}(\varphi^{-1})\varphi(y).$$

So this map is $g^{-1}\mathrm{Jac}(\varphi^{-1})\varphi g$, and the Jacobean is the identity.

Corollary 3.2. *Suppose M^n is closed and has a $\Gamma q(G)$-foliation with G finite. Then there is a principal G-bundle $\pi\colon \overline{M}^n \to M^n$ such that $\overline{M}^n$ is a fiber bundle over the torus T^q.*

Proof. An immediate consequence of Theorems 2.1 and 3.1.

From the above it follows that one can construct manifolds with a $\Gamma q(G)$-foliation in the following way:

Take a manifold M^n with a $\Gamma q(e)$-foliation. Then this foliation is given by $\Omega = \omega_1 \wedge \cdots \wedge \omega_q$ (see Lemma 1.2). Now assume that we have a free G-action with the extra property that for any $g \in G$ we have

$$g^*\omega = \omega \cdot g,$$

where $\omega = (\omega_1, \cdots, \omega_n)$. Then there is a $\Gamma q(G)$-foliation on M^n/G.

If the orbit-manifold M^n/G is open, then the foliation is integrably homotopic to one such that the G-action satisfies the extra property by Theorem 1.4. However if M^n/G is closed, this is no longer true.

Proposition 3.3. *There is a manifold M^n with a $\Gamma q(e)$-foliation and a free G-action such that M^n/G is closed and does not have a $\Gamma q(G)$-foliation.*

Proof. For simplicity let $q = n$, and let Σ^n be a homotopy sphere of dimension n. Consider the connected sum $M^n = T^n \# \Sigma^n$. When Σ^n is not the standard sphere, M^n is P.L.-homeomorphic but not diffeomorphic to T^n; cf. Wall [15]. On the other hand any closed flat Riemannian manifold which is homotopy equivalent to T^n is actually affinely diffeomorphic to T^n (see Wolf [16]). So from Proposition 1.3 we know that M^n does not admit an integrable finite G-structure, when Σ^n is exotic. Nevertheless we can find as principal Z_k-bundle $\pi\colon T^n \to M^n$. First note that the group of homotopy spheres θ_n in dimension $n > 4$ is a finite group by Kervaire-Milnor [10]. Choose an element $\Sigma^n \in \Theta_n$ of order k. Then we can construct the principal Z_k-bundle as follows:

Let $T^n = T^n \# \Sigma^n \# \cdots \# \Sigma^n$ (k copies of Σ^n), and define the free Z_k-action on $T^n \# (\Sigma^n)^k$:

On T^n it is a standard Z_k-action induced from a Z_k-action on the circle S^1, and on $(\Sigma^n)^k$ it is a cyclic permutation.

Theorem 3.4. *If M^n is closed and has a $\Gamma q(G)$-foliation $\mathfrak{F}$ with $G \subset 0(q)$, then the holonomy group G' is finite. In particular Corollary* 3.2 *is still true.*

Proof. The theorem says that $\mathfrak{F}$ is actually a $\Gamma q(G')$-foliation with $G' \subset G$, and G' finite. The proof follows from the ideas in Bieberbach's structure theorem for crystallographic groups (see Wolf [16]). According to Theorem 3.1 we can find a principal G-bundle π: $\overline{M}^n \to M^n$ such that $\overline{M}^n$ has a $\Gamma q(e)$-foliation. Following the proof of Theorem 2.1, $\overline{M}^n$ is a fiber bundle $F \to \overline{M}^n \to T^r \times R^{q-r}$.

Then G acts on $T^r \times R^{q-r}$ as isometries, hence $G \subseteq \Gamma q(0(q))$. As in the proof of Bieberbach's structure theorem we consider $Z^l = G \cap (T^r \times R^{q-r})$ (intersection with the subgroup of translations). Since G preserves this lattice and $0(q)$ is compact, the projection $G \to 0(q)$ has a finite image.

In the special case with $q = n$, only finitely many groups appear as holonomy groups in each dimension n, but any finite group appear for $n \geqslant n_0$ for some n_0 (see Auslander-Kuranishi [2]). q.e.d.

If we drop the condition $G \subset 0(q)$, Theorem 3.4 is no longer true.

Lemma 3.5. *There exists closed manifolds M^n with a $\Gamma q(G)$-foliation but no $\Gamma q(G')$-foliation for finite G'.*

Proof. It clearly suffices to put $q = n$. From the work of Auslander [1] and Auslander-Markus [3] we know that there are infinitely many different homotopy types of closed affine flat manifolds in dimension 3. On the other hand we also know from the work of Bieberbach (see Wolf [16]) that homotopy equivalent closed Riemannian flat manifolds are affinely equivalent, and there are only finitely many in each dimension up to affine equivalence. q.e.d.

If we drop the condition M^n being closed, then Theorem 3.4 is also no longer true.

Lemma 3.6. (a) *There exist open manifolds M^n with a $\Gamma q(G)$-foliation with $G \subset 0(q)$ (even transversely complete) with infinite holonomy group.*

(b) *There exist open manifolds M^n with a $\Gamma q(G)$-foliation with $G \subset 0(q)$ but no $\Gamma q(G')$-foliation for finite G'.*

Proof. (a) This is essentially answered in Wolf [16] together with arguments as in §2 (since the transverse Riemannian metric is complete, the arguments leading to Theorem 2.4, 2.5 and 2.6 apply). It follows that these manifolds M^n are vector bundles over a manifold with a $\Gamma(G')$-foliation and G' finite. Topologically these vector bundles are trivial, but the induced structure group is a discrete subgroup of $\Gamma g'(G')$ for some $q' \leqslant q$, and hence the holonomy group is not finite in general.

(b) By Theorem 1.4 we know that M^n has a $\Gamma q(G)$-foliation if and only if M^n has a q-dimensional subbundle of the tangent bundle with a G-structure, and this can be chosen such that it cannot be reduced to a finite structure. In fact, it can be chosen such that it does not have any finite structure at all (by the recent work of Sullivan-Deligne this requires a manifold M^n not of finite type).

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